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On the Rallison and Acrivos solution for the deformation and burst of a viscous drop in an extensional flow

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In this work we prove that the second-kind Fredholm's integral equations proposed by Rallison & Acrivos to solve the deformation and burst of a viscous drop in an extensional flow, with viscosity ratio λ, possess a unique continuous solution $\boldsymbol{u}(x)$ for any continuous datum $\boldsymbol{F}(x)$ when $0 < \lambda < \infty$. In the original work they could only guarantee, analytically, the solvability of the integral equations in a small neighbourhood of $\lambda = 1$.

1. Introduction

When drops of one fluid are suspended in a second fluid that is caused to shear, the drops will deform and, if the local shear rate is sufficiently large, will break into two or more fragments (for a good literature survey see Rallison 1984).

We consider a viscous drop immersed in a different viscous fluid, with viscosity ratio λ, which has interfacial surface tension γ. The fluid at infinity is made to flow with velocity $u_i^\infty(x) = E_{ij}x_j$ and the drop consequently deforms. The governing equations for the fluid velocity $\boldsymbol{u}$ and pressure p are given by the Stokes' equations:

$$\frac{\partial u_i}{\partial x_i} = 0; \quad \frac{\partial \sigma_{ij}}{\partial x_j} = 0, \tag{1.1}$$

where

$$\sigma_{ij} = \begin{cases} -p\delta_{ij} + \left(\dfrac{\partial u_i}{\partial x_j} + \dfrac{\partial u_j}{\partial x_i}\right) \text{ for } x \in \Omega_{\mathrm{e}}, & (1.2a) \\[2ex] -p\delta_{ij} + \lambda\left(\dfrac{\partial u_i}{\partial x_j} + \dfrac{\partial u_j}{\partial x_i}\right) \text{for } x \in \Omega_{\mathrm{i}}. & (1.2b) \end{cases}$$

Here we have normalized the fluid velocity using the viscosity μ of the carrying fluid; the drop viscosity is $\lambda\mu$. The flow fields have to satisfy the following asymptotic and matching conditions:

$$u_i \to u_i^\infty \quad \text{as } |x| \to \infty \tag{1.3}$$

and

$$[\boldsymbol{u}]_S = 0, \quad [\sigma_{ij} n_i]_S = \gamma n_j \nabla . \boldsymbol{n} \tag{1.4}$$

where $[\ ]_S$ denotes the jump across the surface of the drop S from the outside Ω_{e} to the inside Ω_{i}, $\boldsymbol{n}$ is the outward unit normal and $\nabla \cdot \boldsymbol{n}$ is the surface curvature.

A numerical solution for the deformation between near spheres and slender bodies has been developed by Rallison & Acrivos (1978); using the Green's integral representation formulae for the fluids inside and outside the drop they found a second-kind Fredholm's integral equation for the unknown surface velocity $\boldsymbol{u}(x)$:

$$u_i(\xi)+\frac{2(1-\lambda)}{(1+\lambda)}\int_S K_{ij}(\xi,y)\,u_j(y)\,\mathrm{d}S_y = F_i(\xi) \quad \text{for } \xi\in S, \tag{1.5}$$

with
$$F_i(\xi) = \frac{2}{(1+\lambda)}[E_{ij}\,\xi_j+\int_S J_{ij}(\xi,y)\,n_j(y)\,\boldsymbol{\nabla}\cdot\boldsymbol{n}\,\mathrm{d}S_y].$$

Here
$$\left.\begin{aligned} J_{ij}(x,y) &= -\frac{1}{8\pi}\left(\frac{\delta_{ij}}{r}+\frac{(x_i-y_i)(x_j-y_j)}{r^3}\right)\\ r &= |x-y| \end{aligned}\right\} \tag{1.6}$$

is the fundamental singular solution of Stokes' equations, known as a 'Stokeslet' located at the point y, and

$$\begin{aligned} K_{ij}(x,y) &= \sigma_{ij}(J_{lk}\,e_l)_y\,n_k(y)\\ &= -\frac{3}{4\pi}\frac{(x_i-y_i)(x_j-y_j)(x_k-y_k)}{r^5}n_k(y). \end{aligned} \tag{1.7}$$

When $\lambda = 1$, (1.5) takes the particular simple form

$$u_i(\xi) = E_{ij}\,\xi_j+\int_S J_{ij}(\xi,y)\,n_j(y)\,\boldsymbol{\nabla}\cdot\boldsymbol{n}\,\mathrm{d}S_y \tag{1.8}$$

which in fact is valid at all points x, not just those on S.

It is known that the homogeneous form of (1.5) has just one eigensolution when $\lambda = 0$, and if $\lambda = \infty$ the six rigid-body motions for the drop are all eigensolutions (see Ladyzhenskaya 1969, chap. 3). Therefore from Fredholm's alternative it follows that the integral equation (1.5) does not admit a unique solution at these two poles of the resolvent. Also, it is clear that the resolvent does not have a pole at $\lambda = 1$ and therefore the same will be true in some small neighbourhood around $\lambda = 1$. Rallison & Acrivos conjecture that probably there are no eigensolutions for $0 < \lambda < \infty$, since their numerical solution encountered no difficulties for the values of λ tested in such range. In the next section, we prove, analytically, that the integral equation (1.5) possesses a unique continuous solution $\boldsymbol{u}(x)$ for any continuous datum $\boldsymbol{F}(x)$ when $0 < \lambda < \infty$; in other words the resolvent of (1.5) does not have a pole in this range of λ.

2. Uniqueness of solutions of the integral equation (1.5)

In order to show that the integral equation (1.5) possesses a unique continuous solution $\boldsymbol{u}(x)$ for any continuous datum $\boldsymbol{F}(x)$ when $0 < \lambda < \infty$ it is sufficient, according to Fredholm's alternative, to show that the following homogeneous system (2.1) for $\boldsymbol{\psi}$ admits only a trivial solution in the space of continuous functions when $-1 < \beta < 1$:

$$\psi_i(\xi)-2\beta\int_S K_{ij}(\xi,y)\,\psi_j(y)\,\mathrm{d}S_y = 0 \quad \text{for } \xi\in S, \tag{2.1}$$

where
$$\beta = \frac{(\lambda-1)}{(1+\lambda)}$$

or, equivalently, it is enough to verify that the resolvent of the adjoint homogeneous system to the integral equation (2.1), given below by (2.2), does not have a pole between $\beta = -1$ and $\beta = 1$:

$$\phi_i(\xi) - 2\beta \int_S K_{ji}(y, \xi)\, \phi_j(y)\, \mathrm{d}S_y = 0 \quad \text{for } \xi \in S. \tag{2.2}$$

To prove the above statement, we will follow Goursat's (1964, p. 188) ideas to study the poles of the resolvent of the second-kind Fredholm integral equation resulting from the Dirichlet boundary condition for the classical potential theory.

Let us consider the potential of a single layer

$$V_i(x) = \int_S J_{ij}(x, y)\, \phi_j(y)\, \mathrm{d}S_y, \tag{2.3}$$

where $\boldsymbol{\phi}$ is a non-trivial solution of (2.2). The velocity vector (2.3) together with its corresponding pressure is a well-defined and continuous Stokes flow throughout the entire space (here it is not necessary to consider the existence of the drop, since we are dealing with the solvability of (1.5)), whose surface tension experiences a jump across S, and this jump is given by Ladyzhenskaya (1969 p. 56), as

$$\sigma_{ij}(V)_{\mathrm{i}}\, n_j = \tfrac{1}{2}\phi_i(\xi) - \int_S K_{ji}(y, \xi)\, \phi_j(y)\, \mathrm{d}S_y, \tag{2.4a}$$

$$\sigma_{ij}(V)_{\mathrm{e}}\, n_j = -\tfrac{1}{2}\phi_i(\xi) - \int_S K_{ji}(y, \xi)\, \phi_j(y)\, \mathrm{d}S_y. \tag{2.4b}$$

The above formulae permit the transformation of (2.2) into the form

$$[\sigma_{ij}(V)_{\mathrm{i}}\, n_j - \sigma_{ij}(V)_{\mathrm{e}}\, n_j] + \beta[\sigma_{ij}(V)_{\mathrm{i}}\, n_j + \sigma_{ij}(V)_{\mathrm{e}}\, n_j] = 0 \tag{2.5}$$

for every $\xi \in S$. Here $\sigma_{ij}(V)_{\mathrm{i}}$ is the limiting value of the stress $\sigma_{ij}(V)$ when a point x tends to a point $\xi \in S$ coming from Ω_{i}, and $\sigma_{ij}(V)_{\mathrm{e}}$ is the limiting value of the stress when x tends to ξ coming from Ω_{e}. Equation (2.5) implies that

$$(1+\beta)\, \sigma_{ij}(V)_{\mathrm{i}}\, n_j = (1-\beta)\, \sigma_{ij}(V)_{\mathrm{e}}\, n_j, \quad \xi \in S. \tag{2.6}$$

Let us multiply (2.6) by $V_i^{\mathrm{i}} = V_i^{\mathrm{e}}$ and integrate over S:

$$(1+\beta) \int_S V_i^{\mathrm{i}} \sigma_{ij}(V)_{\mathrm{i}}\, n_j\, \mathrm{d}S = (1-\beta) \int_S V_i^{\mathrm{e}} \sigma_{ij}(V)_{\mathrm{e}}\, n_j\, \mathrm{d}S, \tag{2.7}$$

where V_i^{i} is the limiting value of (2.3) when x tends to $\xi \in S$ coming from Ω_{i}, and V_i^{e} the limiting value when x tends to ξ coming from Ω_{e}, which are identically equal owing to the continuity of a single layer potential across the density carrying surface.

Since (2.3) together with its corresponding pressure is a well-defined continuous Stokes' flow throughout the entire space, whose velocity behaves like $|x|^{-1}$ for large $|x|$ and pressure behaves like $|x|^{-2}$, from Green's first identity for Stokes' equations (see Ladyzhenskaya 1963, p. 57) we obtain the following relations:

$$\int_S \sigma_{ij}(V)_{\mathrm{i}}\, n_j\, V_i^{\mathrm{i}}\, \mathrm{d}S = \frac{1}{2} \int_{\Omega_{\mathrm{i}}} \left(\frac{\partial V_j}{\partial x_i} + \frac{\partial V_i}{\partial x_j} \right)^2 \mathrm{d}x, \tag{2.8a}$$

$$\int_S \sigma_{ij}(V)_{\mathrm{e}}\, n_j\, V_i^{\mathrm{e}}\, \mathrm{d}S = -\frac{1}{2} \int_{\Omega_{\mathrm{e}}} \left(\frac{\partial V_i}{\partial x_j} + \frac{\partial V_j}{\partial x_i} \right)^2 \mathrm{d}x. \tag{2.8b}$$

Hence, in particular, it follows that in (2.7) the integral on the left is non-negative, whilst the integral on the right is non-positive. Therefore, such a relation is impossible if β lies between -1 and $+1$, for the factor $(1+\beta)/(1-\beta)$ is then positive, and the two integrals are of opposite sign. Then, in this case it would be necessary that the two integrals are zero:

$$\int_S \sigma_{ij}(V)_i n_j V_i^i \,\mathrm{d}S = \int_S \sigma_{ij}(V)_e n_j V_i^e \,\mathrm{d}S \equiv 0, \tag{2.9}$$

and from (2.8*a*, *b*) it follows that

$$\frac{\partial V_i}{\partial x_j} + \frac{\partial V_j}{\partial x_i} = 0 \quad \text{for } x \in \Omega \cup \Omega_e, \tag{2.10}$$

a system which is known to have six linearly independent solutions, corresponding to the motion of the fluid as a rigid body. Therefore $V(x, \phi)$ vanishes throughout the space, since such a null field is the only rigid-motion continuous velocity compatible with the asymptotic behaviour of (2.3) at infinity, and hence ϕ is also identically zero. Therefore, for $-1 < \beta < 1$ the integral equation (2.2) admits only the trivial solution, or equivalently the integral equation (1.5) possesses a unique continuous solution $u(x)$ for any continuous datum $F(x)$ when $0 < \lambda < \infty$. Thus we have proved that the deformation and burst of a viscous drop in an extensional flow can be solved by solving Rallison & Acrivos second-kind Fredholm's integral equations for any viscous ratio λ, except the $\lambda = 0$ and $\lambda = \infty$ cases in which the integral equations do not possess a unique solution.

REFERENCES

Goursat, E. 1964 *A Course in Mathematical Analysis*, vol. II, part two. Dover.

Ladyzhenskaya, O. A. 1963 *The Mathematical Theory of Viscous Incompresible Flow*. Gordon and Breach.

Rallison, J. M. & Acrivos, A. 1978 A numerical study of the deformation and burst of a viscous drop in an extensional flow. *J. Fluid Mech.* **89**, 191–200.

Rallison, J. M. 1984 The deformation of small viscous drops and bubbles in shear flow. *Ann. Rev. Fluid Mech.* **16**, 45–66.